THE CAHN-HILLIARD EQUATION AND THE ALLEN-CAHN EQUATION ON MANIFOLDS WITH CONICAL SINGULARITIES

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Abstract. We consider the Cahn-Hilliard equation on a manifold with conical singularities and show the existence of bounded imaginary powers for suitable closed extensions of the bilaplacian. Combining results and methods from singular analysis with a theorem of Clément and Li we then prove the short time solvability of the Cahn-Hilliard equation in L_p-Mellin-Sobolev spaces and obtain the asymptotics of the solution near the conical points.

We deduce, in particular, that regularity is preserved on the smooth part of the manifold and singularities remain confined to the conical points.

We finally show how the Allen-Cahn equation can be treated by simpler considerations. Again we obtain short time solvability and the behavior near the conical points.

1. Introduction

The Cahn-Hilliard equation is a phase-field or diffuse interface equation which is mainly used to model phase separation of a binary mixture, e.g. a two-component alloy, but many other applications are encountered.

In the literature, one finds the equation stated in various forms. We shall consider here the version

$$\partial_t u(t) + \Delta^2 u(t) + \Delta\big(u(t) - u^3(t)\big) = 0, \quad t \in (0,T); \tag{1.1}$$

$$u(0) = u_0, \tag{1.2}$$

where u models the concentration difference of the components. The sets where $u = \pm 1$ correspond to domains of pure phases. The existence of solutions – even global existence – is not an issue since the work of Elliott and Zheng Songmu [8] in 1986 and Caffarelli and Muler [3] in 1995. Our main point of interest is to clarify to what extent the singularities of the underlying space – here a manifold with conical points – are reflected in a short time solution of the equation.

As usual, we model a manifold with conical singularities by a manifold with boundary $\mathbb{B}$ of dimension $n+1$, $n \geq 1$, endowed with a conically degenerate Riemannian metric. On one hand, working on a manifold with boundary simplifies the analysis; on the other hand, the degeneracy of the Riemannian metric entails that geometric operators such as the Laplacian show the typical degeneracy they have on spaces with conic points in Euclidean space.

We measure smoothness in terms of weighted Mellin-Sobolev spaces $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$. Here s is a smoothness index, γ a weight, and $1 < p < \infty$. They coincide with the usual L^p-Sobolev spaces away from the singularities. Close to a conical point, in coordinates (x, y), where x is the distance to the tip and y a tangential variable, one captures differentiability in terms of the operators $x\partial_x$ and ∂_y. For $s = 0$ we obtain an L^p-space with weight $x^{(\frac{n+1}{2}-\gamma)p-1}$. It will serve as the base space for our considerations.

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One of the essential points then is to understand the linearized equation, in particular, the bilaplacian Δ^2 which is the leading order contribution.

As this is a conically degenerate differential operator, a first issue is the choice of a suitable closed extension. Brüning and Seeley [2] first noticed that there is no canonical choice of a closed extension for such operators. Instead one has a minimal and a maximal closed extension. The corresponding domains differ by a finite-dimensional space of functions which are smooth in the tangential variable y and have certain asymptotics in x as $x \to 0^+$, see Lesch [9] or Schrohe and Seiler [10] for more details.

We base our analysis here on that of the Laplacian and choose the domain of Δ^2 accordingly. The closed extensions of the Laplacian have been studied in [10]. Some basic facts are recalled, below. We deduce, in particular, that there exist extensions $\underline{\Delta}$ of the Laplacian for which $c - \underline{\Delta}$ has bounded imaginary powers on our weighted L^p-space for suitably large $c > 0$. We next show that the corresponding result is true for the bilaplacian on an appropriately chosen domain which we determine explicitly in Proposition 3.4. Generically, it is a direct sum of the space $\mathcal{H}_p^{4,4+\gamma}(\mathbb{B})$, which is a weighted space of functions belonging to $H^4_{p,loc}(\mathbb{B}^\circ)$ over the interior $\mathbb{B}^\circ$ of $\mathbb{B}$, and a finite-dimensional space of functions with asymptotics near $x = 0$ as mentioned above. Here, the occurring asymptotics types are of the form x^{-q} or $x^{-q}\log x$, where the exponents q can be determined from the spectrum of the Laplace-Beltrami operator Δ_∂ induced by Δ on the cross-section of the cone.

Our argument then relies on the notion of maximal regularity: Let $X_1 \hookrightarrow X_0$ be Banach spaces and let $B : \mathcal{D}(B) = X_1 \to X_0$ be a closed densely defined linear operator. Assume that $-B$ generates an analytic semigroup. Then the operator B is said to have maximal regularity for the pair (X_1, X_0) and $1 < q < \infty$, if for every v_0 in the interpolation space $X_q = (X_0, X_1)_{1-1/q,q}$ and every $g \in L^q(0, T; X_0)$ there exists a unique solution $v \in L^q(0, T; X_1) \cap W^1_q(0, T; X_0) \cap C([0, T]; X_q)$ of the equation

$$\dot{v} + Bv = g,\ t \in (0, T); \quad v(0) = v_0, \tag{1.3}$$

depending continuously on the data v_0 and g.

It was proven by Dore and Venni, see Theorem 3.2 in [7], that essentially the existence of bounded imaginary powers for B implies maximal regularity, see Theorem 2.3 for details. Replacing v by $e^{ct}v$, it is even sufficient to show that $c + B$ has bounded imaginary powers for large positive c.

A theorem by Clément and Li shows how maximal regularity can be used to establish short time existence of quasilinear equations of the form

$$\partial_t u(t) + A(u(t))u(t) = f(t, u(t)) + g(t),\ t \in (0, T_0); \quad u(0) = u_0 \tag{1.4}$$

in X_0 with domain $\mathcal{D}(A(u(t))) = X_1$, where $T_0 > 0$.

Theorem 1.1. (Clément and Li, [6],Theorem 2.1) *Assume that there exists an open neighborhood U of u_0 in X_q such that $A(u_0)$ has maximal regularity for (X_1, X_0) and q, and that*

(H1) $A \in C^{1-}(U, \mathcal{L}(X_1, X_0))$
(H2) $f \in C^{1-,1-}([0, T_0] \times U, X_0)$,
(H3) $g \in L^q([0, T_0], X_0)$.

Then there exists a $T > 0$ and a unique $u \in L^q(0, T; X_1) \cap W^1_q(0, T; X_0) \cap C([0, T]; X_q)$ solving the equation (1.4) *on* $]0, T[$.

From Theorem 1.1 we deduce the short time existence of solutions to the Cahn-Hilliard equation. In our case, the space X_0 is the weighted L_p-space $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$, while X_1 is the domain of the bilaplacian. The description of u then provides information on the regularity of u and its asymptotics near the conical point. Note that measuring regularity in standard Sobolev spaces is not possible as our manifold is not even C^1-smooth.

We finally turn to the Allen-Cahn equation, a semilinear heat equation of the form

$$\partial_t u(t) - \Delta u(t) = f(u), \quad t \in (0,T); \tag{1.5}$$

$$u(0) = u_0. \tag{1.6}$$

Here, $f : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, which is usually assumed to be of the form $f = F'$ where F has a double well structure (a fact not needed for our arguments). For the extensions $\underline{\Delta}$ of the Laplacian determined above we immediately obtain the existence of a short time solution from Theorem 1.1. Again, the description of the domain provides some asymptotic information.

As both, the Cahn-Hilliard equation and the Allen-Cahn equation are only semilinear, we might have relied on a more elementary approach. The present setting, however, is rather elegant and allows us to make use of earlier work by Coriasco, Schrohe, and Seiler [5]. Moreover, the results for the bilaplacian on conic manifolds which we derive here will be useful later on.

This article is structured as follows: In Section 2 we first introduce the weighted L^p Mellin-Sobolev spaces. We next recall the essential facts about domains and extensions of the Laplacian on manifolds with straight conical singularities. Depending on the dimension we then find suitable extensions $\underline{\Delta}$ for which $c - \underline{\Delta}$ has bounded imaginary powers for suitably large $c > 0$. Section 3 focuses on the description of the domain of the bilaplacian and the proof of maximal regularity for the linear part of the equation. In Section 4 we apply the above theorem by Clément and Li. We find that there is a delicate interplay between the choice of the weight (and hence the extension) and the conditions of the theorem. The choices depend on the dimension. Of course, the two-dimensional case, where the phases may be considered as films on a surface with conical singularities, is of greatest practical interest. The Allen-Cahn equation is addressed in Section 5.

2. Notation and Preliminary Results

2.1. **Bounded imaginary powers.** Following Amann [1], Sections 4.6 and 4.7, we give the following two definitions:

Definition 2.1. *Let X be a Banach space, $K \geq 1$ and $\theta \in [0, \pi[$. We denote by $\mathcal{P}(K,\theta)$ the class of all closed, densely defined linear operators A in X such that*

$$(1+|z|)\,\|(A+z)^{-1}\| \leq K \quad \text{for all } z \in S_\theta = \{z \in \mathbb{C} : |\arg z| \leq \theta\} \cup \{0\} \subset \rho(-A).$$

In particular, we let $S_0 = \mathbb{R}^+ \cup \{0\}$ and write $\mathcal{P}(\theta) = \bigcup_K \mathcal{P}(K,\theta)$.

Definition 2.2. *Let X be a Banach space, $M \geq 1$ and $\phi \geq 0$. We say that a linear operator A in X has bounded imaginary powers with angle ϕ and write $A \in \mathcal{BIP}(M,\phi)$, provided $A \in \bigcup_\theta \mathcal{P}(\theta)$, the imaginary powers A^{it} are defined for $t \in \mathbb{R}$, and we have the estimate*

$$\|A^{it}\|_{\mathcal{L}(X)} \leq M e^{\phi|t|}, \ t \in \mathbb{R}.$$

We let $\mathcal{BIP}(\phi) = \bigcup_M \mathcal{BIP}(M,\phi)$.

The importance of bounded imaginary powers is illustrated by the aforementioned result by Dore and Venni [7], Theorem 3.2.

Theorem 2.3. *Let X be a ζ-convex Banach space and $A \in \mathcal{P}(0) \cap \mathcal{BIP}(\phi)$ for some $0 \leq \phi < \frac{\pi}{2}$. Then A has maximal regularity for the pair $(\mathcal{D}(A), X)$.*

2.2. **The Laplacian on Mellin-Sobolev spaces over a manifold with conical singularities.** Let $\mathbb{B}$ be an $n+1$ dimensional smooth compact manifold with boundary $\partial\mathbb{B}$. We fix a collar neighborhood diffeomorphic to $[0,1)\times\partial\mathbb{B}$, where we denote coordinates by (x,y), $x\in[0,1)$, $y\in\mathbb{B}$.

We assume that $\mathbb{B}$ is endowed with a Riemannian metric which, in the above neighborhood takes the degenerate form $g=dx^2+x^2h$, where h is a Riemannian metric on $\mathbb{B}$. The associated Laplacian then is a second order cone differential operator. It is of the form

$$\Delta=\frac{1}{x^2}\Big((x\partial_x)^2+(n-1)x\partial_x+\Delta_\partial\Big) \tag{2.7}$$

near the boundary, where Δ_∂ is the Laplacian on $\partial\mathbb{B}$ induced by g.

By a cut-off function (near $\partial\mathbb{B}$) we mean a smooth non-negative function ω with $\omega\equiv 1$ near ∂B and $\omega\equiv 0$ outside the collar neighborhood of the boundary.

Definition 2.4. *Let $k\in\mathbb{N}_0$, γ in $\mathbb{R}$ and $1\le p<\infty$. By $\mathcal{H}_p^{k,\gamma}(\mathbb{B})$ we denote the space of all functions u on $\mathbb{B}$ such that for each cut-off function ω we have $(1-\omega)u\in H_p^k(\mathbb{B})$ and*

$$x^{\frac{n+1}{2}-\gamma}(x\partial_x)^j\partial_y^\alpha(\omega u)(x,y)\in L^p\left(\frac{dx}{x}dy\right),\quad j+|\alpha|\le k.$$

There are various ways of extending the definition in order to obtain Banach spaces $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ for all $s\in\mathbb{R}$. One of the simplest ways, cf. [4], is to define the map

$$\mathcal{S}_\gamma: C_c^\infty(\mathbb{R}^{n+1})\to C_c^\infty(\mathbb{R}^{n+1}),\qquad v(t,y)\mapsto e^{(\frac{n+1}{2}-\gamma)t}v(e^{-t},y).$$

Moreover, let $\kappa_j: U_j\subseteq\partial\mathbb{B}\to\mathbb{R}^n$, $j=1,\dots,N$, be a covering of $\partial\mathbb{B}$ by coordinate charts and $\{\varphi_j\}$ a subordinate partition of unity. Then $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is the space of all distributions such that

$$\|u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})} = \sum_{j=1}^N \|\mathcal{S}_\gamma(1\times\kappa_j)_*(\omega\varphi_j u)\|_{H_p^s(\mathbb{R}^{1+n})}+\|(1-\omega)u)\|_{H_p^s(\mathbb{B})} \tag{2.8}$$

is defined and finite. Here, ω is a (fixed) cut-off function and $*$ refers to the push-forward of distributions. Up to equivalence of norms, this construction is independent of the choice of ω and the κ_j. Clearly, $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ is a UMD space and hence ζ-convex.

Corollary 2.5. *Let $1\le p<\infty$ and $s>(n+1)/p$. Then a function u in $\mathcal{H}_p^{s,\gamma}(\mathbb{B})$ is continuous on $\mathbb{B}^\circ$, and, near $\partial\mathbb{B}$, we have*

$$|u(x,y)|\le cx^{\gamma-(n+1)/2}\|u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})}$$

for a constant $c>0$.

Proof. Continuity on $\mathbb{B}^\circ$ follows from the usual Sobolev embedding theorem, noting that $\mathcal{H}_p^{s,\gamma}(\mathbb{B})\hookrightarrow H_{p,loc}^s(\mathbb{B})$. Near the boundary, we deduce from (2.8) and the trace theorem that for fixed $t\in\mathbb{R}$,

$$e^{((n+1)/2-\gamma)t}\|u(e^{-t},\cdot)\|_{B_{p,p}^{s-1/p}(\partial\mathbb{B})}\le c\|u\|_{\mathcal{H}_p^{s,\gamma}(\mathbb{B})}.$$

Letting $x=e^{-t}$ we then obtain the assertion from the fact that the Besov space $B_{p,p}^{s-1/p}(\partial\mathbb{B})$ embeds into the Sobolev space $H_p^{s-1/p-\varepsilon}(\partial\mathbb{B})$ for every $\varepsilon>0$ and the Sobolev embedding theorem. $\square$

2.3. **Closed extensions of the Laplace operator.** As pointed out in the introduction, the choice of a suitable closed extension of Δ is of central importance. For the convenience of the reader we recall here the basic facts following [10], where more details can be found.

In the analysis of conically degenerate (pseudo-)differential operators, the so-called conormal symbol plays an important role, see e.g. Schulze [11] for an exhaustive treatment. Also here, the first step is the analysis of the conormal symbol $\sigma_M(\Delta)$ of Δ, i.e. the operator-valued function

$$\sigma_M(\Delta) : \mathbb{C} \to \mathcal{L}(H_p^s(\partial\mathbb{B}), H_p^{s-2}(\partial\mathbb{B})) \ \text{ given by } \quad \sigma_M(\Delta)(z) = z^2 - (n-1)z + \Delta_\partial.$$

We are interested in the values of z where $\sigma_M(\Delta)$ is not invertible. For this, the precise choice of s is not essential. We denote by $0 = \lambda_0 > \lambda_1 > \dots$ the eigenvalues of Δ_∂ and by $E_0, E_1, \dots$ the corresponding eigenspaces. Moreover, let $\pi_j \in \mathcal{L}(L_2(\partial\mathbb{B}))$ be the orthogonal projection onto E_j; it extends to $L^p(\partial\mathbb{B})$ for $1 < p < \infty$: For an L^2-orthonormal basis $\{e_{j1}, \dots, e_{jm}\}$ of E_j we let $\pi_j(v) = \sum_{k=1}^m \langle v, e_{jk}\rangle e_{jk}$.

The *non*-bijectivity points of $\sigma_M(\Delta)$ are the points $z = q_j^+$ and $z = q_j^-$ with

$$q_j^\pm = \tfrac{n-1}{2} \pm \sqrt{\left(\tfrac{n-1}{2}\right)^2 - \lambda_j}, \qquad j \in \mathbb{N}_0. \tag{2.9}$$

Note the symmetry $q_j^+ = (n-1) - q_j^-$. It is straightforward to see that

$$(z^2 - (n-1)z + \Delta_\partial)^{-1} = \sum_{j=0}^\infty \frac{1}{(z - q_j^+)(z - q_j^-)} \pi_j. \tag{2.10}$$

In fact, this is a pseudodifferential operator which clearly is inverse to $\sigma_M(\Delta)$ on $L^2(\partial\mathbb{B})$. Thus it also is the inverse on $H_p^s(\partial\mathbb{B})$ for arbitrary s and $1 < p < \infty$, since the span of the eigenfunctions of Δ_∂ is dense in these spaces.

Hence, in case $\dim\mathbb{B} \neq 2$, where the $q_j^\pm$ are all different, the inverse to $\sigma_M(\Delta)$ has only simple poles in the points $q_j^\pm$. For $\dim\mathbb{B} = 2$ the poles at $q_j^\pm$, $j \neq 0$ are simple, while there is a double pole at $q_0^+ = q_0^- = 0$.

With $q_j^\pm$, $j \neq 0$, we associate the function space

$$\mathcal{E}_{q_j^\pm} = \omega\, x^{-q_j^\pm} \otimes E_j = \{\omega(x)\, x^{-q_j^\pm}\, e(y) : e \in E_j\}, \ j \in \mathbb{N}.$$

For $j = 0$ we let

$$\mathcal{E}_{q_0^\pm} = \begin{cases} \omega \otimes E_0 + \omega \log x \otimes E_0 & \dim\mathbb{B} = 2 \\ \omega\, x^{q_0^\pm} \otimes E_0 & \dim\mathbb{B} \neq 2 \end{cases}. \tag{2.11}$$

For later use note that Δ maps the spaces $\mathcal{E}_{q_j^\pm}$ to $C_c^\infty(\mathbb{B}^\circ)$.

Furthermore, we introduce the sets I_γ, $\gamma \in \mathbb{R}$, by

$$I_\gamma = \{q_j^\pm : j \in \mathbb{N}_0\} \cap \left]\tfrac{n+1}{2} - \gamma - 2, \tfrac{n+1}{2} - \gamma\right[.$$

The following is Proposition 5.1 in [10]:

Proposition 2.6. *The domain of the maximal extension of Δ in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ is*

$$\mathcal{D}(\Delta_{\max}) = \mathcal{D}(\Delta_{\min}) \oplus \bigoplus_{q_j^\pm \in I_\gamma} \mathcal{E}_{q_j^\pm}.$$

In case $q_j^\pm \neq \frac{n+1}{2} - \gamma - 2$ for all j, the minimal domain is $\mathcal{D}(\Delta_{\min}) = \mathcal{H}_p^{2,\gamma+2}(\mathbb{B})$.

Corollary 2.7. *The domains of the closed extensions of Δ are the sets of the form $\mathcal{D}(\Delta_{\min}) \oplus \mathcal{E}$, where $\mathcal{E}$ is any subspace of $\oplus_{q_j^\pm \in I_\gamma} \mathcal{E}_{q_j^\pm}$.*

Definition 2.8. *Given a subspace* $\underline{\mathcal{E}}_{q_j^\pm}$ *of* $\mathcal{E}_{q_j^\pm}$*, we define the space* $\underline{\mathcal{E}}^\perp_{q_j^\pm}$ *as follows:*

i) *If either* $q_j^\pm \neq 0$ *or* $\dim \mathbb{B} \neq 2$*, there exists a unique subspace* $\underline{E}_j \subseteq E_j$ *such that* $\underline{\mathcal{E}}_{q_j^\pm} = \underline{E}_j \otimes \omega\, x^{-q_j^\pm}$*. Then we set*
$$\underline{\mathcal{E}}^\perp_{q_j^\pm} = \omega\, x^{-q_j^\mp} \otimes \underline{E}_j^\perp,$$
where $\underline{E}_j^\perp$ *is the orthogonal complement of* $\underline{E}_j$ *in* E_j *with respect to the* $L^2(\partial\mathbb{B})$*-scalar product.*

ii) *For* $\dim\mathbb{B} = 2$ *and* $q_0^\pm = 0$ *define* $\underline{\mathcal{E}}_0^\perp = \{0\}$ *if* $\underline{\mathcal{E}}_0 = \mathcal{E}_0$*,* $\underline{\mathcal{E}}_0^\perp = \mathcal{E}_0$ *if* $\underline{\mathcal{E}}_0 = \{0\}$*, and* $\underline{\mathcal{E}}_0^\perp = \underline{\mathcal{E}}_0$ *if* $\underline{\mathcal{E}}_0 = \omega \otimes E_0$*.*

Note that $\underline{\mathcal{E}}^\perp_{q_j^\pm}$ *is a subspace of* $\mathcal{E}_{q_j^\mp}$*. For* $\dim\mathbb{B} = 2$ *we let* $\mathcal{E}_{00} = \omega \otimes E_0$*.*

We now confine ourselves to extensions $\underline{\Delta}$ with domains
$$\mathcal{D}(\underline{\Delta}) = \mathcal{D}(\Delta_{\min}) \oplus \bigoplus_{q_j^\pm \in I_\gamma} \underline{\mathcal{E}}_{q_j^\pm} \subseteq \mathcal{H}_p^{0,\gamma}(\mathbb{B})$$
chosen according to the following rules:

(i) If $q_j^\pm \in I_\gamma \cap I_{-\gamma}$, then $\underline{\mathcal{E}}^\perp_{q_j^\pm} = \underline{\mathcal{E}}_{(n-1)-q_j^\pm}$.
(ii) If $\gamma \geq 0$ and $q_j^\pm \in I_\gamma \setminus I_{-\gamma}$, then $\underline{\mathcal{E}}_{q_j^\pm} = \mathcal{E}_{q_j^\pm}$.
(iii) If $\gamma \leq 0$ and $q_j^\pm \in I_\gamma \setminus I_{-\gamma}$, then $\underline{\mathcal{E}}_{q_j^\pm} = \{0\}$. [1]

In particular, $\mathcal{D}_p^\gamma(\underline{\Delta}) = \mathcal{D}_p^\gamma(\Delta_{\max})$ if $\gamma \geq 1$ and $\mathcal{D}_p^\gamma(\underline{\Delta}) = \mathcal{D}_p^\gamma(\Delta_{\min})$ if $\gamma \leq -1$.

Theorem 2.9. *Let* $\theta \in [0,\pi[$*, and* $\phi > 0$*.*

(a) *For* $|\gamma| < \dim\mathbb{B}/2$ *and* $|\gamma| < 2$*, let* $\underline{\Delta}$ *be an extension with domain chosen as above. Then* $c - \underline{\Delta} \in \mathcal{P}(\theta) \cap \mathcal{BIP}(\phi)$ *for suitably large* $c > 0$*.*
(b) *For* $\dim\mathbb{B} \geq 4$ *and* $|\gamma| < \dim\mathbb{B}/2$ *let* $\underline{\Delta} = \Delta_{\min}$ *for* $\gamma \leq 0$ *and* $\underline{\Delta} = \Delta_{\max}$ *for* $\gamma > 0$*. Then* $c - \underline{\Delta} \in \mathcal{P}(\theta) \cap \mathcal{BIP}(\phi)$ *for suitably large* $c > 0$*.*

Proof. (a) For $\dim\mathbb{B} \leq 3$ this is [10, Theorem 5.7] combined with [10, Theorem 4.3]. Inspection shows that the proof of [10, Theorem 5.7] extends to higher dimensions provided $|\gamma| < 2$.
(b) is [10, Theorem 5.6] combined with [10, Theorem 4.3]. □

Remark 2.10. For arbitrary dimension of $\mathbb{B}$, the part (a) of Theorem 2.9 extends to the case where $|\gamma| < \dim\mathbb{B}/2$ for an extension $\underline{\Delta}$ satisfying the rules (i), (ii) and (iii) above. This follows by iterating the argument given in the proof of [10, Theorem 5.7] and using that the interval $]0, n-1[$ contains none of $q_j^\pm$.

Remark 2.11. An extension $\underline{\Delta}$ in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$, induces an unbounded operator in $L^p([0,T], \mathcal{H}_p^{0,\gamma}(\mathbb{B}))$ by the relation $(\underline{\Delta}u)(t) = \underline{\Delta}(u(t))$. We denote it again by $\underline{\Delta}$.

3. The Linearized Problem

We recall that the gradient associated to the metric g, $\nabla : C^\infty(\mathbb{B}^\circ) \to \Gamma^\infty(\mathbb{B}^\circ, T\mathbb{B}^\circ)$ is defined by
$$\nabla u = \mathrm{grad} u = \sum_{ij} g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial}{\partial x^j},$$

[1] We have corrected in (iii) the order of I_γ and $I_{-\gamma}$ which was misstated in [10].

where $(x^1, \dots, x^{n+1})$ are local coordinates and $(g^{ij}) = (g_{ij})^{-1}$ is the inverse to the matrix defining g in these coordinates. Near the boundary, $g^{-1} = dx^2 + x^{-2}h^{-1}$ with the notation introduced in Section 2.2. If $T\mathbb{B}^\circ$ is equipped with the Riemannian inner product $(\cdot,\cdot)_g$ given by g, then

$$\Delta u^3 = 3u^2\Delta u - 6u \sum g^{ij} \frac{\partial u}{\partial x^i}\frac{\partial u}{\partial x^j} = 3u^2\Delta u - 6u(\nabla u, \nabla u)_g.$$

In coordinates $(x, y^1, \dots, y^n)$ near the boundary,

$$(\nabla u, \nabla v)_g = \frac{1}{x^2}((x\partial_x u)(x\partial_x v) + \sum_{i,j=1}^{n} h^{ij}(y)\partial_{y^i} u \partial_{y^j} v). \tag{3.12}$$

This allows us to write Equation (1.1) as

$$\partial_t u + A(u)u = F(u), \quad u(0) = u_0 \tag{3.13}$$

with

$$A(v)u = \Delta^2 u + \Delta u - 3v^2\Delta u \text{ and } F(u) = -6u(\nabla u, \nabla u)_g. \tag{3.14}$$

In order to find a suitable domain for the unbounded operator $A(v)$, we next study the Laplacian.

3.1. **The choice of an extension of Δ.** We proceed to define an extension $\underline{\Delta}$ of the Laplacian on $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ for any $1 < p < \infty$ satisfying the assumptions of Theorem 2.9. Our choice will depend on the dimension. We abbreviate

$$\bar{\varepsilon} = -q_1^- > 0. \tag{3.15}$$

Note that $\bar{\varepsilon}$ actually depends on n and the spectrum of Δ_∂.

Proposition 3.1. *The assumptions of Theorem 2.9 are fulfilled for the choice of extensions outlined in Sections 3.1.1-3.1.3, below.*

3.1.1. *The two-dimensional case.* For $\dim \mathbb{B} = n + 1 = 2$ we pick a weight

$$-1 < \gamma < \max\{-1 + \bar{\varepsilon}, 1\}.$$

This guarantees that $\frac{n+1}{2} - \gamma - 2$ coincides with none of the $q_j^\pm$ and hence that the minimal domain is $\mathcal{H}_p^{2,2+\gamma}(\mathbb{B})$ by Proposition 2.6. Moreover, $I_\gamma \cap I_{-\gamma} = \{q_0\} = \{0\}$. We therefore choose

$$\mathcal{D}(\underline{\Delta}) = \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}) \oplus \mathcal{E}_{00},$$

cf. Definition 2.8. The domain therefore consists of bounded functions only. Note, moreover, that $\mathcal{E}_{00} \subseteq \mathcal{H}_p^{\infty,1-\delta}(\mathbb{B})$ for every $\delta > 0$.

3.1.2. *The three-dimensional case.* For $\dim \mathbb{B} = 3$ we choose

$$-\frac{1}{2} < \gamma < \max\left\{-\frac{1}{2} + \bar{\varepsilon}, \frac{3}{2}\right\}.$$

Then $\frac{n+1}{2} - \gamma - 2$ coincides with none of the $q_j^\pm$, and the minimal domain is $\mathcal{H}_p^{2,2+\gamma}(\mathbb{B})$ by Proposition 2.6. The intersection $I_\gamma \cap I_{-\gamma}$ equals $\{0, 1\}$ for $\gamma < \frac{1}{2}$, and it is empty for $\frac{1}{2} \le \gamma < \frac{3}{2}$. According to Theorem 2.9 we choose

$$\mathcal{D}(\underline{\Delta}) = \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}) \oplus \mathcal{E}_0, \tag{3.16}$$

where $\mathcal{E}_0$ is the full asymptotics space associated with $q_0^- = 0$.

3.1.3. *Higher dimensions.* Next assume $4 \leq \dim \mathbb{B}$ and choose

$$\frac{n-3}{2} < \gamma < \max\left\{\frac{n-3}{2} + \bar{\varepsilon}, \frac{n+1}{2}\right\}.$$

Then, again $\frac{n+1}{2} - \gamma - 2$ does not coincide with any $q_j^{\pm}$, and the minimal domain is $\mathcal{H}_p^{2,2+\gamma}(\mathbb{B})$ by Proposition 2.6. The intersection $I_\gamma \cap I_{-\gamma}$ is empty. Thus, according to Theorem 2.9 or Remark 2.10, we choose

$$\mathcal{D}(\underline{\Delta}) = \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}) \oplus \mathcal{E}_0, \tag{3.17}$$

where again $\mathcal{E}_0$ is the full asymptotics space of $q_0^- = 0$.

3.2. **The domain of $\underline{\Delta}^2$.** We choose the extension of the bilaplacian induced by our choice of the extension $\underline{\Delta}$, namely

$$\mathcal{D}(\underline{\Delta}^2) = \{u \in \mathcal{D}(\underline{\Delta}) : \Delta u \in \mathcal{D}(\underline{\Delta})\}.$$

According to (2.13) in [10], its conormal symbol is the function

$$\sigma_M(\Delta^2)(z) = \sigma_M(\Delta)(z+2)\sigma_M(\Delta).$$

A formula for the inverse follows from (2.10) and the orthogonality of the projections π_j:

$$\begin{aligned}\sigma_M(\Delta^2)(z)^{-1} &= \sum_{j,k=0}^{\infty} \frac{1}{(z-q_j^+)(z-q_j^-)(z+2-q_k^+)(z+2-q_k^-)}\pi_j\pi_k \\ &= \sum_{j=0}^{\infty} \frac{1}{(z-q_j^+)(z-q_j^-)(z+2-q_j^+)(z+2-q_j^-)}\pi_j.\end{aligned}$$

In fact, this is the inverse on $L^2(\partial\mathbb{B})$. As it is a pseudodifferential operator, it extends/restricts to $H_p^s(\partial\mathbb{B})$ for all $1 < p < \infty, s \in \mathbb{R}$. Clearly, we have poles at the points $z = q_j^{\pm}$ and $z = q_j^{\pm} - 2$. We denote the collection of all these points by $\mathcal{Q}$. We obtain:

Lemma 3.2. (a) *If $\dim \mathbb{B} = 2$, then we have at least two double poles, namely at $z = 0$ and $z = -2$. An additional double pole occurs if $q_j^+ - 2 = q_j^-$ for some j. This requires $\lambda_j = -1$ for some j, so that $z = -1$.*

(b) *If $\dim \mathbb{B} = 3$, then a double pole can only occur if $q_j^+ - 2 = q_j^-$ for some j. As this is precisely the case if $\lambda_j = -3/4$, this pole will be in $z = -1/2$.*

(c) *If $\dim \mathbb{B} = 4$, then we have a double pole at $z = 0$, since then $q_0^+ - 2 = 0 = q_0^-$.*

(d) *For $\dim \mathbb{B} \geq 5$ all poles are simple.*

Remark 3.3. For the analysis of the bilaplacian it is desirable to choose γ such that the line $\{\operatorname{Re} z = \frac{n+1}{2} - \gamma - 4\}$ does not intersect $\mathcal{Q}$, for then the minimal domain is

$$\mathcal{D}(\Delta^2_{\min}) = \mathcal{H}_p^{4,\gamma+4}(\mathbb{B}).$$

In case $\mathcal{Q}$ intersects this line, we have

$$\mathcal{D}(\Delta^2_{\min}) = \{u \in \bigcap_{\varepsilon>0} \mathcal{H}_p^{4,4+\gamma-\varepsilon}(\mathbb{B}) : \Delta^2 u \in \mathcal{H}_p^{0,\gamma}(\mathbb{B})\}.$$

In particular,

$$\mathcal{H}_p^{4,4+\gamma}(\mathbb{B}) \subseteq \mathcal{D}(\Delta^2_{\min}) \subseteq \mathcal{H}_p^{4,4+\gamma-\varepsilon}(\mathbb{B}) \quad \text{for all } \varepsilon > 0. \tag{3.18}$$

See [10, Proposition 2.3] for details.

For a pole $\rho \in \mathcal{Q}$ of order k we denote by $\tilde{\mathcal{E}}_\rho$ the asymptotics space associated to this pole; it is determined by Equation (2.11) in [10] and of the form

$$\tilde{\mathcal{E}}_\rho = \operatorname{span}\{x^\rho \log^l x\, \omega(x)e(y) : l = 0, \dots k-1, e \in \tilde{E}_\rho\}, \tag{3.19}$$

where $\tilde{E}_\rho$ is a finite-dimensional subspace of $C^\infty(\partial\mathbb{B})$ consisting of eigenfunctions of Δ_∂. Note that in our case k can only take the values 1 and 2.

Now we know on one hand that, for $0 \neq e \in C^\infty(\partial\mathbb{B})$,

$$x^{-\rho} \log^l x\, \omega(x)e \in \mathcal{H}_p^{s,\gamma}(\mathbb{B}) \text{ if and only if } \operatorname{Re} \rho < \frac{n+1}{2} - \gamma;$$

on the other hand, for a pole in $\rho \in \mathcal{Q}$ of order k, $l < k$, and $e \in \tilde{E}_q$,

$$\Delta(x^{-\rho} \log^l x\, \omega(x)e) \in C_c^\infty(\mathbb{B}^\circ) \subseteq \mathcal{H}_p^{\infty,\infty}(\mathbb{B}).$$

Proposition 3.4. *We define the interval*

$$J = J(n,\gamma) = \left]\frac{n+1}{2} - \gamma - 4, \frac{n+1}{2} - \gamma - 2\right[.$$

With the choices made in Sections 3.1.1-3.1.3 we have

(a) *For* $\dim \mathbb{B} = 2$ *and the extension* $\underline{\Delta}$ *in 3.1.1 we have*

$$\mathcal{D}(\underline{\Delta}^2) = \mathcal{D}(\Delta^2_{\min}) \oplus \bigoplus_{\rho \in J} \tilde{\mathcal{E}}_\rho \oplus \mathcal{E}_{00}.$$

(b) *For* $\dim \mathbb{B} \geq 3$ *and the extension* $\underline{\Delta}$ *in 3.1.2 or 3.1.3 we have*

$$\mathcal{D}(\underline{\Delta}^2) = \mathcal{D}(\Delta^2_{\min}) \oplus \bigoplus_{\rho \in J} \tilde{\mathcal{E}}_\rho \oplus \mathcal{E}_{0}.$$

Corollary 3.5. We can now describe the domain of the bilaplacian explicitly, using (3.19). Let J be the interval introduced in Proposition 3.4.

(a) For $\dim \mathbb{B} = 2$ and the extension in 3.4(a), where $-1 < \gamma \leq 0$, the interval $J =]-3-\gamma, -1-\gamma[$ will contain the double pole in $z = 0$ and the possible double pole in $z = -1$, but not that in $z = 2$.
(b) For $\dim \mathbb{B} = 3$ and the extension in 3.4(b), the interval J will contain the only possible double pole at $z = -1/2$ if and only if $q_1^+ < 3/2$.
(c) For $\dim \mathbb{B} = 4$ and the extension in 3.4(b), there is a double pole in $z = 0$ which is not contained in $J =]-\gamma-2, -\gamma[$.
(d) For $\dim \mathbb{B} > 4$ no double poles arise.

3.3. **Embedding the interpolation space X_q.** We shall apply the theorem of Clément and Li with the choices $X_0 = \mathcal{H}_p^{0,\gamma}(\mathbb{B})$ and $X_1 = \mathcal{D}(\underline{\Delta}^2)$ for $2 < q < \infty$. We next look for a suitable embedding of the interpolation space X_q. For arbitrary η with $1/2 < \eta < 1 - 1/q$ we have

$$X_q := (X_0, X_1)_{1-\frac{1}{q},q} \hookrightarrow [X_0, X_1]_\eta = [\mathcal{H}_p^{0,\gamma}(\mathbb{B}), \mathcal{D}(\underline{\Delta}^2)]_\eta = [\mathcal{D}((c-\underline{\Delta})^0), \mathcal{D}((c-\underline{\Delta})^2)]_\eta, \tag{3.20}$$

for suitably large $c > 0$. Since $c - \underline{\Delta} \in \mathcal{BIP}(\phi)$ for any $\phi > 0$ and sufficiently large $c > 0$, we apply (I.2.9.8) in [1] and obtain

$$[\mathcal{D}((c-\underline{\Delta})^0), \mathcal{D}((c-\underline{\Delta})^2)]_\eta \hookrightarrow \mathcal{D}((c-\underline{\Delta})^{(1-\eta)0+2\eta}) = \mathcal{D}((c-\underline{\Delta})^{2\eta}). \tag{3.21}$$

As $\eta > 1/2$, we have $2\eta = 1+\vartheta$ for some $\vartheta > 0$. We apply once more (I.2.9.8) in [1] and use the fact that $\mathcal{D}(\underline{\Delta}) \subseteq \mathcal{H}_p^{2,\gamma+\varepsilon_0}(\mathbb{B})$ and $\mathcal{D}(\underline{\Delta}^2) \subseteq \mathcal{H}_p^{4,\gamma+\varepsilon_1}(\mathbb{B})$ for suitable $\varepsilon_0, \varepsilon_1 > 0$ and $0 < \vartheta' < \vartheta$:

$$\begin{aligned} &\mathcal{D}((c-\underline{\Delta})^{2\eta}) = \mathcal{D}((c-\underline{\Delta})^{1+\vartheta}) \\ = \;& [\mathcal{D}(c-\underline{\Delta}), \mathcal{D}((c-\underline{\Delta})^2)]_\vartheta \hookrightarrow [\mathcal{H}_p^{2,\gamma+\varepsilon_0}(\mathbb{B}), \mathcal{H}_p^{4,\gamma+\varepsilon_1}(\mathbb{B})]_\vartheta \\ \hookrightarrow \;& (\mathcal{H}_p^{2,\gamma+\varepsilon_0}(\mathbb{B}), \mathcal{H}_p^{4,\gamma+\varepsilon_1}(\mathbb{B}))_{\vartheta',p}. \end{aligned} \tag{3.22}$$

Next we use Lemma 5.4 in [4] to conclude that, for arbitrary $\delta_0, \delta_1 > 0$, we have

$$\begin{aligned} &(\mathcal{H}_p^{2,\gamma+\varepsilon_0}(\mathbb{B}), \mathcal{H}_p^{4,\gamma+\varepsilon_1}(\mathbb{B}))_{\vartheta',p} \\ &\quad \hookrightarrow \; \mathcal{H}_p^{4\vartheta'+2(1-\vartheta')-\delta_0,\gamma+\vartheta'\varepsilon_1+(1-\vartheta')\varepsilon_0-\delta_1}(\mathbb{B}) = \mathcal{H}_p^{2+2\vartheta'-\delta_0,\gamma+\vartheta'\varepsilon_1+(1-\vartheta')\varepsilon_0-\delta_1}(\mathbb{B}). \end{aligned}$$

Summing up, we see that

$$X_q \hookrightarrow \mathcal{D}(\underline{\Delta}) \cap \mathcal{H}_p^{2+2\vartheta-\delta_0,\gamma+\vartheta\varepsilon_1+(1-\vartheta)\varepsilon_0-\delta_1}(\mathbb{B})$$

for every ϑ with $0 < \vartheta < 1 - 2/q$.

For the extensions in Proposition 3.1 we have $\mathcal{D}(\underline{\Delta}) \subseteq L^\infty(\mathbb{B})$ and hence $X_q \subseteq L^\infty(\mathbb{B})$. By (3.14),

$$\mathcal{D}(A(v)) = \mathcal{D}(\underline{\Delta}^2), \quad v \in X_q.$$

3.4. **Bounded imaginary powers.** The following observation might be well-known. As we did not find a reference, we include a proof:

Lemma 3.6. *Let E be a Banach space and $A \in \mathcal{P}(\theta)$ with $\theta \geq \pi/2$. Then $A^2 \in \mathcal{P}(\tilde{\theta})$ for $\tilde{\theta} = 2\theta - \pi$ and $(A^2)^z = A^{2z}$ for $z \in \mathbb{C}$.*

Proof. In view of the fact that A^{-2z} and $(A^2)^{-z}$ are holomorphic operator families for $\mathrm{Re}(z) > 0$ we can confine ourselves to the case $0 < \mathrm{Re}(z) < \frac{1}{2}$.

Let $A \in \mathcal{P}(K, \theta)$ for $K \geq 1$. For $s \geq 0$ the resolvent formula implies that

$$(A^2+\lambda)^{-1} = (A - i\sqrt{\lambda})^{-1}(A+i\sqrt{\lambda})^{-1} = \frac{1}{2i\sqrt{\lambda}}\left((A-i\sqrt{\lambda})^{-1} - (A+i\sqrt{\lambda})^{-1}\right). \tag{3.23}$$

We note that $\arg(\pm i\sqrt{\lambda}) = \frac{1}{2}\arg\lambda \pm \frac{1}{2}\pi$. Thus, for $\lambda \in S_{\tilde{\theta}}$ with $|\lambda|$ away of zero,

$$(1+|\lambda|)\|(A^2+\lambda)^{-1}\| \leq \frac{1+|\lambda|}{2|\sqrt{\lambda}|}\left(\|(A-i\sqrt{\lambda})^{-1}\| + \|(A+i\sqrt{\lambda})^{-1}\|\right) \leq \tilde{K},$$

for some $\tilde{K} > 0$, and hence $A^2 \in \mathcal{P}(K', \tilde{\theta})$ for some $K' \geq 1$. Following Amann, cf. (III.4.6.9) in [1], we let

$$(A^2)^{-z} = \frac{\sin \pi z}{\pi}\int_0^{+\infty} u^{-z}(A^2+u)^{-1}du, \quad 0 < \mathrm{Re}(z) < \frac{1}{2}.$$

By (3.23) we find that

$$
\begin{aligned}
(A^2)^{-z} = &\frac{\sin \pi z}{\pi}\int_0^{+\infty}\frac{u^{-z}}{2i\sqrt{u}}\big((A-i\sqrt{u})^{-1}-(A+i\sqrt{u})^{-1}\big)\,du\\
= &\frac{\sin \pi z}{\pi}\int_0^{-i\infty}(-\lambda^2)^{-z}(A+\lambda)^{-1}d\lambda+\frac{\sin \pi z}{\pi}\int_0^{+i\infty}(-\lambda^2)^{-z}(A+\lambda)^{-1}d\lambda\\
= &-\frac{e^{i\pi z}-e^{-i\pi z}}{2\pi i}\int_{-i\infty}^{0}(e^{i\pi}\lambda^2)^{-z}(A+\lambda)^{-1}d\lambda+\frac{e^{i\pi z}-e^{-i\pi z}}{2\pi i}\int_0^{+i\infty}(e^{-i\pi}\lambda^2)^{-z}(A+\lambda)^{-1}d\lambda\\
= &\frac{1}{2\pi i}\int_{-i\infty}^{0}(e^{i\pi}\lambda)^{-2z}(A+\lambda)^{-1}d\lambda+\frac{1}{2\pi i}\int_0^{+i\infty}(e^{-i\pi}\lambda)^{-2z}(A+\lambda)^{-1}d\lambda-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\lambda^{-2z}(A+\lambda)^{-1}d\lambda\\
= &\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}(-\lambda)^{-2z}(A+\lambda)^{-1}d\lambda = A^{-2z},
\end{aligned}
$$

where we have used the fact that

$$\int_{-i\infty}^{+i\infty}\lambda^{-2z}(A+\lambda)^{-1}d\lambda=0,$$

since λ^{-2z} is holomorphic for $\mathrm{Re}(\lambda)>0$. □

In the following proposition, $\underline{\Delta}$ denotes the dilation invariant extension of the Laplacian defined in Section 3.1 and $A(u_0)$ is the operator defined in (3.14) with the choice $\underline{\Delta}$ for the Laplacian.

Proposition 3.7. *For every choice of $u_0\in L^\infty(\mathbb{B})$), $\phi>0$, and $\theta\in[0,\pi[$, the operator $A(u_0)+c_0I$, considered as an unbounded operator in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ with domain $\mathcal{D}(\underline{\Delta}^2)$ belongs to $\mathcal{P}(\theta)\cap\mathcal{BIP}(\phi)$ for all sufficiently large $c_0>0$.*

Proof. By possibly increasing θ we may assume that $\max\{\pi-\theta,\phi\}=\phi$. Theorem 2.9 asserts that $A=c-\underline{\Delta}$ belongs to $\mathcal{P}(K,(\theta+\pi)/2)\cap\mathcal{BIP}(\phi/2)$ with suitable K, provided c is large. Now Lemma 3.6 implies that $A^2\in\mathcal{P}(\theta)$ $(A^2)^z=A^{2z}$ for $\mathrm{Re}(z)<0$ and hence that $A^2\in\mathcal{BIP}(\phi)$.

Moreover, $A^2+\mu\in\mathcal{P}(\theta)$ for $\mu\geq0$. By Corollary III.4.8.6 in [1], $A^2+\mu\in\mathcal{BIP}(\max\{\pi-\theta,\phi\})=\mathcal{BIP}(\phi)$. In order to obtain $\mathcal{BIP}$ for $A(u_0)$, we apply a perturbation result, namely Theorem III.4.8.5 in [1] for the perturbation $B=2c\underline{\Delta}$. In accordance with the notation used there, we denote by $\Gamma(k,\psi)$ the negatively oriented boundary of

$$\{|\arg(z)|\leq\psi_k\}\cup\{|z|\leq 1/2k\},\ \text{ where }\ \psi_k=\min\left\{\frac{\pi+\psi}{2},\arcsin\frac{1}{2k}\right\}.$$

We will have $(A^2+\mu)+B=\underline{\Delta}^2+c^2+\mu\in\mathcal{BIP}(\phi)$, if we can show that for suitable $0<\beta<1$ and $K_1\geq1$

(i) $\|B(A^2+\mu+\lambda)^{-1}\|\leq\beta$ for all $\lambda\in\Gamma\cup S(\theta)$, where $\Gamma=\Gamma((1-\beta)^{-1}K_1,\theta)$
(ii) $(A^2+\mu+\lambda)^{-1}B(A^2+\mu+\lambda)^{-1}\in L^1(\Gamma,d\lambda,\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B})))$.

Concerning (i):

$$\begin{aligned}
&\|B(A^2+\mu+\lambda)^{-1}\| = 2c\|\underline{\Delta}((c-\underline{\Delta})^2+\mu+\lambda)^{-1}\|\\
&= 2c\|\underline{\Delta}(-\underline{\Delta}+c-i\sqrt{\mu+\lambda})^{-1}(-\underline{\Delta}+c+i\sqrt{\mu+\lambda})^{-1}\|\\
&\leq 2c\|(-\underline{\Delta}+c-i\sqrt{\mu+\lambda})^{-1}\|\|(-\underline{\Delta}+c+i\sqrt{\mu+\lambda}-(c+i\sqrt{\mu+\lambda}))(-\underline{\Delta}+c+i\sqrt{\mu+\lambda})^{-1}\|\\
&\leq \frac{2cK}{1+|\sqrt{\mu+\lambda}|}\|I-(c+i\sqrt{\mu+\lambda})(-\underline{\Delta}+c+i\sqrt{\mu+\lambda})^{-1}\|\\
&\leq \frac{2cK}{1+|\sqrt{\mu+\lambda}|}\left(1+\frac{K|c+i\sqrt{\mu+\lambda}|}{1+|\sqrt{\mu+\lambda}|}\right)
\end{aligned}$$

The last expression can be estimated by β provided μ is taken sufficiently large.

Concerning (ii):

$$\begin{aligned}
&\|(A^2+\mu+\lambda)^{-1}B(A^2+\mu+\lambda)^{-1}\| = \|((-\underline{\Delta}+c)^2+\mu+\lambda)^{-1}2c\underline{\Delta}((-\underline{\Delta}+c)^2+\mu+\lambda)^{-1}\|\\
&= 2c\|(-\underline{\Delta}+c-i\sqrt{\mu+\lambda})^{-1}(-\underline{\Delta}+c+i\sqrt{\mu+\lambda})^{-1}\Big(-\underline{\Delta}+c-i\sqrt{\mu+\lambda}-(c-i\sqrt{\mu+\lambda})\Big)\\
&\quad(-\underline{\Delta}+c-i\sqrt{\mu+\lambda})^{-1}(-\underline{\Delta}+c+i\sqrt{\mu+\lambda})^{-1}\|\\
&\leq 2c\Big(\frac{K}{1+|\sqrt{\mu+\lambda}|}\Big)^3\Big(1+\frac{K|c-i\sqrt{\mu+\lambda}|}{1+|\sqrt{\mu+\lambda}|}\Big) = O(|\mu+\lambda|^{-\frac{3}{2}})
\end{aligned}$$

so that also (ii) holds for sufficiently large μ.

We write $\tilde{c}=c^2+\mu$ for μ as above, and apply once more Theorem III.4.8.5 in [1], now with $\underline{\Delta}^2+\tilde{c}$ in the role of A and $h\underline{\Delta}$ in the role of B for an arbitrary $h\in L^\infty(\mathbb{B})$. Obviously $\mathcal{D}(h\underline{\Delta})\supseteq \mathrm{Dom}(\underline{\Delta}^2+\tilde{c})$. From (3.23) we see that $\underline{\Delta}^2+\tilde{c}\in\mathcal{P}(\tilde{K},\theta)$ for some $\tilde{K}\geq 1$. We now have to check the analogs of conditions (i) and (ii) above. Similarly as before, we note concerning (i) that, by possibly increasing μ,

$$\begin{aligned}
&\|h\underline{\Delta}\left(\underline{\Delta}^2+\tilde{c}+\lambda\right)^{-1}\| \leq \|h\|_\infty\|\underline{\Delta}\left(-\underline{\Delta}+i\sqrt{\tilde{c}+\lambda}\right)^{-1}\left(-\underline{\Delta}-i\sqrt{\tilde{c}+\lambda}\right)^{-1}\|\\
&\leq \|h\|_\infty\left\|\left(I-i\sqrt{\tilde{c}+\lambda}\left(-\underline{\Delta}+i\sqrt{\tilde{c}+\lambda}\right)^{-1}\right)\left(-\underline{\Delta}-i\sqrt{\tilde{c}+\lambda}\right)^{-1}\right\|\\
&\leq \|h\|_\infty\left(1+\frac{K|\sqrt{\tilde{c}+\lambda}|}{1+|c-i\sqrt{\tilde{c}+\lambda}|}\right)\frac{K}{1+|c+i\sqrt{\tilde{c}+\lambda}|}\leq\beta,
\end{aligned}$$

for λ in $\Gamma((1-\beta)^{-1}K'',\theta)\cup S_\theta$, with $K''\geq\tilde{K}$ sufficiently large. Moreover, concerning (ii),

$$\begin{aligned}
&\|(\underline{\Delta}^2+\tilde{c}+\lambda)^{-1}h\underline{\Delta}(\underline{\Delta}^2+\tilde{c}+\lambda)^{-1}\|\\
&\leq \|(\underline{\Delta}^2+\tilde{c}+\lambda)^{-1}\|\|h\|_\infty\|\underline{\Delta}(\underline{\Delta}^2+\tilde{c}+\lambda)^{-1}\|\\
&\leq \|h\|_\infty\|(-\underline{\Delta}-i\sqrt{\tilde{c}+\lambda})^{-1}(-\underline{\Delta}+i\sqrt{\tilde{c}+\lambda})^{-1}\|\\
&\quad\left\|\big(I+i\sqrt{\tilde{c}+\lambda}(-\underline{\Delta}-i\sqrt{\tilde{c}+\lambda})^{-1}\big)(-\underline{\Delta}+i\sqrt{\tilde{c}+\lambda})^{-1}\right\|\\
&\leq \|h\|_\infty\left(\frac{K}{1+|c-i\sqrt{\tilde{c}+\lambda}|}\right)^2\left(1+\frac{K|\sqrt{\tilde{c}+\lambda}|}{1+|c+i\sqrt{\tilde{c}+\lambda}|}\right)\frac{K}{1+|c+i\sqrt{\tilde{c}+\lambda}|}=O(\lambda^{-\frac{3}{2}}),
\end{aligned}$$

so that the operator is $L^1(\big(\Gamma((1-\beta)^{-1}K'',\theta),ds,\mathcal{L}(\mathcal{H}_p^{0,\gamma}(\mathbb{B}))\big)$ if we possibly increase μ further. Choosing $h=1-3u_0^2$ we get the result. $\square$

4. The Nonlinear Equation in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$

4.1. **The two-dimensional case.** Let $\dim \mathbb{B} = 2$ and $\underline{\Delta}^2$ the extension of the bilaplacian determined in Proposition 3.4(a).

Lemma 4.1. *Let (x,y) be local coordinates near $\partial\mathbb{B}$. For $u \in X_q$, with $q > 2$, we have $x\partial_x u$ and $\partial_y u$ in $\mathcal{H}_p^{1+\varepsilon,2+\gamma}(\mathbb{B})$ for sufficiently small $\varepsilon > 0$. They are therefore L^∞-functions whenever $p \geq 2$.*

Proof. We recall from Section 3.3 that X_q embeds into the interpolation space $[\mathcal{D}(\underline{\Delta}), \mathcal{D}((\underline{\Delta})^2)]_\vartheta$ for some $\vartheta > 0$. Since the functions in E_0 are locally constant, the functions in $\mathcal{E}_{00}$ are also locally constant near the boundary. Taking any derivative will result in a function which equals zero near the boundary. Hence, the operators $x\partial_x$ and ∂_y map $\mathcal{D}(\underline{\Delta})$ to $\mathcal{H}_p^{1,2+\gamma}(\mathbb{B})$ and as they preserve the asymptotics of the functions in $\tilde{\mathcal{E}}_\rho$, with $\rho \in J$, they map $\mathcal{D}(\underline{\Delta}^2)$ to $\mathcal{H}_p^{3,2+\gamma+\varepsilon}(\mathbb{B})$ for some $\varepsilon > 0$. Thus, they map X_q to $[\mathcal{H}_p^{1,2+\gamma}(\mathbb{B}), \mathcal{H}_p^{3,2+\gamma+\varepsilon}(\mathbb{B})]_\vartheta$ for some $\vartheta > 0$. The latter space embeds to $\mathcal{H}_p^{1+\delta,2+\gamma+\delta}(\mathbb{B})$ for some $\delta > 0$, which is a subset of $L^\infty(\mathbb{B})$. □

Corollary 4.2. *We infer from (3.12) that for $u, v \in X_q$,*

$$\begin{aligned}
&\|(\nabla u, \nabla v)_g\|_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \\
&\quad \leq c_1 \quad \max\{\|x\partial_x u\|_{L^\infty}, \|\partial_y u\|_{L^\infty(\mathbb{B})}\} \\
&\qquad\qquad \times \max\{\|x^{-2}(x\partial_x v)\|_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})}, \|x^{-2}\partial_y v\|_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})} \leq c_2 \|u\|_{X_q}\|v\|_{X_q}
\end{aligned}$$

with suitable constants c_1, c_2.

Theorem 4.3. *Let $\dim \mathbb{B} = 2$, $q > 2$ and $p \geq 2$. Given any $u_0 \in X_q$, there exists a $T > 0$ and a unique solution*

$$u \in L^q([0,T], \mathcal{D}(\underline{\Delta}^2)) \cap W_q^1([0,T], \mathcal{H}_p^{0,\gamma}(\mathbb{B})) \cap C([0,T], X_q)$$

solving Equation (1.1) on $]0,T[$ with initial condition (1.2).

Proof. Given $\phi > 0$ we know from Proposition 3.7 that $A(u_0) + c_0 I$ has $\mathcal{BIP}(\phi)$ provided $c_0 > 0$ is large. Hence we have maximal regularity by Dore and Venni's theorem. Next let us check conditions (H1) and (H2) in Clément and Li's theorem; note that (H3) is not required for this particular equation. Let U be a bounded neighborhood of u_0 in X_q. We noted in Section 3.1 that U consists of bounded functions.

Concerning (H1): Let $u_1, u_2 \in U$. Then

$$\begin{aligned}
&\|A(u_1) - A(u_2)\|_{\mathcal{L}(X_1,X_0)} = 3\|(u_1^2 - u_2^2)\underline{\Delta}\|_{\mathcal{L}(X_1,X_0)} \leq c\|(u_1^2 - u_2^2)I\|_{\mathcal{L}(\mathcal{D}(\underline{\Delta}),X_0)} \\
&\quad \leq \quad c_1\|(u_1 - u_2)(u_1 + u_2)\|_\infty \leq c_2(\|u_1\|_{X_q} + \|u_2\|_{X_q})\|u_1 - u_2\|_{X_q} \leq c_3\|u_1 - u_2\|_{X_q}
\end{aligned}$$

for suitable constants c_1, c_2 and c_3, where the last inequality is a consequence of the boundedness of U.

Concerning (H2), we argue that

$$\begin{aligned}
&\|F(u_1) - F(u_2)\|_{X_0} = \|6u_1(\nabla u_1, \nabla u_1)_g - 6u_2(\nabla u_2, \nabla u_2)_g\|_{X_0} \\
&\quad \leq \quad 6\|u_1(\nabla u_1, \nabla u_1)_g - u_2(\nabla u_1, \nabla u_1)_g\|_{X_0} + 6\|u_2(\nabla u_1, \nabla u_1)_g - u_2(\nabla u_1, \nabla u_2)_g\|_{X_0} \\
&\qquad\quad +6\|u_2(\nabla u_1, \nabla u_2)_g - u_2(\nabla u_2, \nabla u_2)_g\|_{X_0} \\
&\quad \leq \quad 6(\|u_1 - u_2\|_\infty\|(\nabla u_1, \nabla u_1)_g\|_{X_0} + 6\|u_2\|_\infty\|(\nabla u_1, \nabla(u_1 - u_2))_g\|_{X_0} \\
&\qquad\quad +6\|u_2\|_\infty\|(\nabla u_2, \nabla(u_1 - u_2))_g\|_{X_0}.
\end{aligned}$$

According to Lemma 4.1, we can estimate the right hand side by $c\|u_1 - u_2\|_{X_q}(\|u_1\|_{X_q} + \|u_2\|_{X_q})^2$, which is bounded in view of the boundedness of U. □

4.2. **The higher-dimensional case.** Let $\dim \mathbb{B} \geq 3$, $q > 2$, $p \geq n+1$ and $\underline{\Delta}^2$ the extension determined in Proposition 3.4(b). As $z = 0$ is a simple pole for the inverted Mellin symbol of the Laplacian, $\mathcal{E}_0$ consists of bounded functions which are locally constant near the boundary. Hence $\mathcal{D}(\underline{\Delta})$ embeds into $L^\infty(\mathbb{B})$ and so does $\mathcal{D}(\underline{\Delta}^2) \subseteq \mathcal{D}(\underline{\Delta})$. By interpolation $X_q \hookrightarrow L^\infty(\mathbb{B})$. We have the following analogue of Lemma 4.1:

Lemma 4.4. *Let $(x, y^1, ..., y^n)$ be local coordinates near $\partial\mathbb{B}$, $q > 2$ and $p \geq n+1$. Then $x\partial_x$ and ∂_{y^j}, $j \in \{1, ..., n\}$, are bounded maps from X_q to $\mathcal{H}_p^{1+\varepsilon, 2+\gamma}(\mathbb{B}) \hookrightarrow L^\infty(\mathbb{B})$ for sufficiently small $\varepsilon > 0$.*

Then, similarly to Theorem 4.3, we prove

Theorem 4.5. *Let $\dim \mathbb{B} \geq 3$, $q > 2$, $p \geq n+1$ and $\underline{\Delta}^2$ the extension determined in Proposition 3.4(b). Given any $u_0 \in X_q$, there exists a $T > 0$ and a unique solution in*

$$u \in L^q([0,T], \mathcal{D}(\underline{\Delta}^2)) \cap W_q^1([0,T], \mathcal{H}_p^{0,\gamma}(\mathbb{B})) \cap C([0,T], X_q)$$

solving Equation (1.1) *on* $]0, T[$ *with initial condition* (1.2).

Proof. The fact that u_0 is in L^∞ yields maximal regularity for $A(u_0) + \tilde{c}$ by Proposition 3.7 and Dore and Venni's theorem for large $\tilde{c}$. So we only have to check conditions (H1) and (H2) in Clément and Li's theorem. Let U be a bounded neighborhood of u_0 in X_q and $\tilde{x}$ a function which equals x near $\partial\mathbb{B}$, is strictly positive on $\mathbb{B}^\circ$ and is $\equiv 1$ outside a neighborhood of $\partial\mathbb{B}$.

Concerning (H1): Let $u_1, u_2 \in U$. Then

$$\begin{aligned}
\|A(u_1) - A(u_2)\|_{\mathcal{L}(X_1, X_0)} &= 3\|(u_1^2 - u_2^2)\underline{\Delta}\|_{\mathcal{L}(X_1, X_0)} \leq c\|(u_1^2 - u_2^2)I\|_{\mathcal{L}(\mathcal{D}(\underline{\Delta}), X_0)} \\
&\leq c\|(u_1 + u_2)(u_1 - u_2)I\|_{\mathcal{L}(\mathcal{D}(\underline{\Delta}), \mathcal{H}_p^{0,\gamma}(\mathbb{B}))} \leq c_1\|u_1 + u_2\|_\infty \|u_1 - u_2\|_\infty \\
&\leq c_2(\|u_1\|_{X_q} + \|u_2\|_{X_q})\|u_1 - u_2\|_{X_q} \leq c_3\|u_1 - u_2\|_{X_q}
\end{aligned}$$

for suitable constants c, c_1, c_2, and c_3, where the last inequality is a consequence of the boundedness of U.

Concerning (H2), we first deduce from Lemma 4.4 and Corollary 2.5 that for $u, v \in X_q$, we have near $\partial\mathbb{B}$,

$$\begin{aligned}
|x^2(\nabla u, \nabla v)_g| &= |(x\partial_x u)(x\partial_x v) + \sum h^{ij}(\partial_{y^i} u)(\partial_{y^j} v)| \\
&\leq c_4 x^{2(2+\gamma-\frac{n+1}{2})}\|u\|_{X_q}\|v\|_{X_q},
\end{aligned}$$

for some constant c_4. Hence we can estimate

$$\begin{aligned}
\|F(u_1) - F(u_2)\|_{X_0} &= \|6u_1(\nabla u_1, \nabla u_1)_g - 6u_2(\nabla u_2, \nabla u_2)_g\|_{X_0} \\
&\leq 6\|u_1(\nabla u_1, \nabla u_1)_g - u_2(\nabla u_1, \nabla u_1)_g\|_{X_0} \\
&\quad +6\|u_2(\nabla u_1, \nabla u_1)_g - u_2(\nabla u_1, \nabla u_2)_g\|_{X_0} + 6\|u_2(\nabla u_1, \nabla u_2)_g \\
&\quad -u_2(\nabla u_2, \nabla u_2)_g\|_{X_0} \\
&\leq c_5\Big(\|\tilde{x}^{2(1+\gamma-\frac{n+1}{2})}(u_1 - u_2)\|_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})}\|u_1\|_{X_q}\|u_2\|_{X_q} \\
&\quad +\|\tilde{x}^{2(1+\gamma-\frac{n+1}{2})}u_2\|_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})}(\|u_1\|_{X_q} + \|u_2\|_{X_q})\|u_1 - u_2\|_{X_q}\Big) \\
&\leq c_5\Big(\|\tilde{x}^{2(1+\gamma-\frac{n+1}{2})}\|_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})}\|u_1 - u_2\|_\infty\|u_1\|_{X_q}\|u_2\|_{X_q} \\
&\quad +\|\tilde{x}^{2(1+\gamma-\frac{n+1}{2})}\|_{\mathcal{H}_p^{0,\gamma}(\mathbb{B})}\|u_2\|_\infty(\|u_1\|_{X_q} + \|u_2\|_{X_q})\|u_1 - u_2\|_{X_q}\Big) \\
&\leq c_6\Big(\|u_1\|_{X_q}\|u_2\|_{X_q} + \|u_2\|_{X_q}(\|u_1\|_{X_q} + \|u_2\|_{X_q})\Big)\|u_1 - u_2\|_{X_q}
\end{aligned}$$

with suitable constants c_5, c_6, using the fact that U is bounded in X_q and $\gamma > \frac{n-3}{2}$. $\square$

5. The Allen-Cahn Equation

We will now prove the existence of short time solutions to the Allen-Cahn equation (1.5), with the help of Theorem 1.1. We choose $X_0 = \mathcal{H}_p^{0,\gamma}(\mathbb{B})$ and $X_1 = \mathcal{D}(\underline{\Delta})$ with one of the extensions determined in Theorem 2.9. Clearly $-\underline{\Delta} + c \in \mathcal{BIP}(\phi)$ for any $\phi > 0$, with $c > 0$ sufficiently large, and the operator $A = \underline{\Delta}$ has maximal regularity for (X_1, X_0) and any q, since it is constant as a function of the interpolation space. Moreover, by the last observation, the condition (H1) of Theorem 1.1 is satisfied.

If we take any open set $U \in X_q$ and $u_1, u_2 \in U$, then we infer from the Lipschitz continuity of f that

$$\|f(u_2) - f(u_2)\|_{X_0} \leq c_f \|u_2 - u_2\|_{X_0} \leq c_f \|u_2 - u_2\|_{X_q},$$

for some constant c_f. Hence, the condition (H2) of Theorem 1.1 is also satisfied and we get the following result

Theorem 5.1. *Let $\underline{\Delta}$ be a closed extension of the Laplace operator in $\mathcal{H}_p^{0,\gamma}(\mathbb{B})$ as in Theorem 2.9. Then, for any*

$$u_0 \in X_q = (\mathcal{H}_p^{0,\gamma}(\mathbb{B}), \mathcal{D}(\underline{\Delta}))_{1-\frac{1}{q},q},$$

with $q \in (1, \infty)$, there exists a $T > 0$ such that the problem (1.5), (1.6) *admits a unique solution*

$$u \in L^q([0,T], \mathcal{D}(\underline{\Delta})) \cap W_q^1([0,T], \mathcal{H}_p^{0,\gamma}(\mathbb{B})) \cap C([0,T], X_q).$$

Remark 5.2. If $\dim \mathbb{B} \geq 3$, by restricting the weight to $\gamma \in \,]-\frac{n+1}{2}, \min\{n-3, 0\}]$ with $\gamma \neq \frac{n+1}{2} - q_j^{\pm} - 2$, we can choose the extension $\mathcal{D}(\underline{\Delta}) = \mathcal{D}(\Delta_{\min}) = \mathcal{H}_p^{2,2+\gamma}(\mathbb{B})$. By Lemma 5.4 in [4] we then find that

$$X_q = (X_0, X_1)_{1-\frac{1}{q},q} \hookrightarrow (\mathcal{H}_p^{0,\gamma}(\mathbb{B}), \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}))_{1-\frac{1}{q}-\delta,p} \hookrightarrow \mathcal{H}_p^{2(1-\frac{1}{q})-3\delta, 2(1-\frac{1}{q})+\gamma-3\delta}(\mathbb{B})$$

for any $\delta > 0$. Moreover, for $q = p \leq 2$, Lemma 5.4 in [4] even shows that, for arbitrary $\delta > 0$,

$$X_p = (X_0, X_1)_{1-\frac{1}{p},p} = (\mathcal{H}_p^{0,\gamma}(\mathbb{B}), \mathcal{H}_p^{2,2+\gamma}(\mathbb{B}))_{1-\frac{1}{p},p} \hookrightarrow \mathcal{H}_p^{2(1-\frac{1}{p}), 2(1-\frac{1}{p})+\gamma-\delta}(\mathbb{B}).$$

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